Design of Isotropic Lightweight Material Structures

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We present a structural optimization approach for the design of isotropic lightweight structures. We apply a combination of mathematical optimization and a variant of the inverse homogenization method which can be found in [1].

Problem setting. Let Ω ⊂ ℝⁿ be a Lipschitz-domain containing an elastic body with prescribed values at the boundary ∂Ω and the Sobolev spaces Wᵐ,p(Ω), 0 ≤ m, 1 ≤ p ≤ ∞ with their standard norms.

We study the optimization problem

\[
\min_{\rho \in \mathcal{U}_{ad}} J(E^H) + \text{additional constraints (1)}
\]

where the design variable (or pseudo density) \(\rho\) is restricted to the set of admissible designs

\[
\mathcal{U}_{ad} = \left\{ \rho \in W^{1,\infty} : \rho_{\text{min}} \leq \rho \leq 1, \quad \left| \frac{\partial \rho}{\partial x_i} \right| \leq c \quad (i = 1, 2) \text{ a. e. in } \Omega, \quad \int \rho \, dx \leq V_0, \quad g(\rho) \leq 0 \right\},
\]

and \(g(\rho)\) is a greyness constraint

\[
g(\rho) = \int \rho \, dx - c \cdot \int \rho^p \, dx, \quad p > 1, c > 1.
\]

The slope constraint in \(\mathcal{U}_{ad}\) guarantees compactness in the space of continuous functions w. r. t. the sup-norm and thus existence of solutions, see [2].

Types of constraints. For the goal of designing isotropic materials with extreme properties we either minimize the Poisson’s ratio \(\nu\) or maximize the elasticity modulus \(Y\) and introduce additional constraints.

The first constraint we apply is an isotropy constraint for the homogenized tensor \(E^H\). Due to symmetry, the conditions for a 2D tensor can be formulated as follows:

\[
E_{11} - E_{22} = 0 \quad E_{11} - E_{12} - 2E_{33} = 0 \quad E_{23} = 0.
\]

Analogously, a set of formulas can be obtained for the three dimensional case and for other material classes like orthotropic materials. In contrast to the approach of approximating isotropic material tensors using tracking type functionals, in this formulation the optimal design is always isotropic.

The second constraint is to prescribe intervals for the elasticity modulus \(Y\) and the Poisson’s ratio \(\nu\)

\[
Y_{\text{min}} \leq Y \leq Y_{\text{max}} \quad \text{and} \quad \nu_{\text{min}} \leq \nu \leq \nu_{\text{max}}
\]
which are of particular importance when the Poisson’s ratio is minimized in order to avoid materials with arbitrarily low elasticity modulus.

**Applications.** We investigate two specific applications both of which arise from concrete engineering applications. Firstly, we want to design lightweight (ceramic) foams with prescribed porosity and maximized stiffness:

\[
\max_{\rho \in \mathcal{U}_{ad}} Y \\
Y_{\text{min}} \leq Y \leq Y_{\text{max}} \quad \text{and} \quad E^H \text{ is isotropic}
\]

\((6)\)

The second application is the design of isotropic 3D auxetic structures (i.e. structures with a negative Poisson’s ratio) which should be realized as metallic foams:

\[
\min_{\rho \in \mathcal{U}_{ad}} \nu \\
Y_{\text{min}} \leq Y \leq Y_{\text{max}} \quad \text{and} \quad E^H \text{ is isotropic.}
\]

\((7)\)

Preliminary results are depicted in Fig. 1.

![Figure 1: (a) optimal design for a foam with maximal elasticity modulus. (b) 3D auxetic structure.](image)

**References**
