Shape derivatives of boundary integral operators in electromagnetic scattering

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Introduction. Consider the scattering of time-harmonic electromagnetic waves by a bounded obstacle $\Omega$ in $\mathbb{R}^3$ with smooth and simply connected boundary $\Gamma$ filled with an homogeneous dielectric material. Boundary integral equations are an efficient method to solve such problems for low and high frequencies. Optimal shape design with the modulus of the far field pattern of the dielectric scattering problem as goal is of practical interest in important fields of applied physics, as for example telecommunication systems and radars. The utilization of shape optimization methods requires the analysis of the dependency of the solution on the shape of the dielectric scatterer. An explicit form of the shape derivatives is required in view of their implementation in shape optimization algorithms such as gradient methods or Newton’s method. Following ideas of [7, 8, 9], we develop a complete shape (or Gâteaux) differentiability analysis of the solution to the dielectric scattering problem using an integral representation [2]. The analysis uses results on traces and electromagnetic potentials [6] and on calculus of variations [4].

Shape derivative analysis. We denote the electric permittivity and the magnetic permeability by $\epsilon_i$, $\mu_i$ in $\Omega$ and by $\epsilon_e$, $\mu_e$ in $\Omega_e = \mathbb{R}^3 \setminus \Omega$. We set $\kappa_i = \omega \sqrt{\epsilon_i \mu_i}$ and $\kappa_e = \omega \sqrt{\epsilon_e \mu_e}$ where $\omega$ is the frequency. Consider an incident wave $E^{inc} \in H_{loc}(\text{curl}, \mathbb{R}^3)$ satisfying $\text{curl}\text{curl}E^{inc} - \kappa_i^2E^{inc} = 0$ in a bounded domain including $\Omega$ and assume that it is a fixed data. We introduce small perturbations of the form $\Gamma_r = (1 + r)\Gamma$ where $r$ is an element of the space of smooth vector functions $C^\infty(\Gamma, \mathbb{R}^3)$. The functions $r$ are chosen small enough in order that $\Gamma_r$ is a smooth and simply connected boundary of a domain $\Omega_r$. We want to study the Gâteaux differentiability of the application mapping $r$ to the solution $(E^i(r), E^e(r))$ in $H(\text{curl}, \Omega_r) \times H_{loc}(\text{curl}, \Omega_r^e)$ to the Maxwell’s equations:

\[
\begin{align*}
\text{curl}\text{curl}E^i(r) - \kappa_i^2E^i(r) &= 0 \quad \text{in } \Omega_r, \\
\text{curl}\text{curl}E^e(r) - \kappa_e^2E^e(r) &= 0 \quad \text{in } \Omega_r^e,
\end{align*}
\]

satisfying the transmission conditions on $\Gamma_r$:

\[
\begin{align*}
n_r \times E^i(r) - n_r \times E^{inc}(r) &= 0, \\
\mu_i^{-1}n_r \times \text{curl}E^i(r) - \mu_e^{-1}n_r \times \text{curl}E^{inc}(r) &= 0,
\end{align*}
\]

where $E^{inc}(r) = E^i(r) + E^{inc}$ and $n_r$ is the outer unit normal vector to $\Gamma_r$ and such that $E^i(r)$ satisfy the Silver-Müller condition. The use of boundary integral equation methods leads us to study the Gâteaux differentiability properties of applications mapping $r$ to one of the potential operators defined by

\[
\begin{align*}
\Psi_{E^i}(r)j_r &= \frac{1}{\kappa} \int_{\Gamma_r} \text{curl}(G(\kappa, |\cdot - y_r|)j_r(y_r))d\sigma(y_r), \\
\Psi_{E^e}(r)j_r &= \int_{\Gamma_r} G(\kappa, |\cdot - y_r|)j_r(y_r)d\sigma(y_r),
\end{align*}
\]

and of applications mapping $r$ to one of the boundary integral operators defined by

\[
\begin{align*}
C_{\kappa}(r)j_r &= \int_{\Gamma_r} n_r \times \text{curl}\{G(\kappa, |\cdot - y_r|)j_r(y_r)\}d\sigma(y_r), \\
M_{\kappa}(r)j_r &= \int_{\Gamma_r} n_r \times \text{curl}\{G(\kappa, |\cdot - y_r|)j_r(y_r)\}d\sigma(y_r).
\end{align*}
\]
consist in using the Helmholtz decomposition of this Hilbert space $H^1$ divergence is in of tangential vector fields whose components are in the Sobolev space $H^2$. We face two main difficulties: On one hand, to be able to construct shape derivatives of the solution it is imperative to prove that the derivatives are bounded operators between the same spaces as the boundary integral operators themselves. On the other hand, the very definition of shape differentiability of operators defined on $H^{1/2}(\Gamma_r)$ poses non-trivial problems. Our approach consist in using the Helmholtz decomposition of this Hilbert space [1]

$$TH^{-1/2}(\text{div}_r, \Gamma_r) = \nabla_TH^{1/2}(\Gamma_r) \oplus \text{curl}_rH^{1/2}(\Gamma_r).$$

We introduce a new invertible operator $P_r$:

$$TH^{-1/2}(\text{div}_r, \Gamma_r) \rightarrow TH^{-1/2}(\text{div}_r, \Gamma)$$

$$\nabla_{\Gamma_r} p_r + \text{curl}_{\Gamma_r} q_r \rightarrow \nabla_{\Gamma} \tau_r p_r + \text{curl}_{\Gamma} \tau_r q_r,$$

where $\tau_r p_r = p_r \circ (1 + r)$. Inserting the identity $I_{TH^{-1/2}(\text{div}_r, \Gamma_r)} = P_r^{-1}P_r$, between each operator in the integral representation of $E'(r)$ and $E''(r)$ we then have to study the Gâteaux differentiability of the applications mapping $r$ to the operators $\Psi_{E4}(r)P_r^{-1}$, $\Psi_{M_4}(r)P_r^{-1}$, $P_rC_4(r)P_r^{-1}$ and $P_rM_4(r)P_r^{-1}$ acting on the fixed space $TH^{-1/2}(\text{div}_r, \Gamma)$. We base their analysis on the study of scalar integral operators with pseudo-homogeneous kernel in Sobolev spaces (see [6], p. 176), but we then have to study the Gâteaux differentiability of numerous surface differential operators. We prove that the electromagnetic boundary integral operators are infinitely Gâteaux differentiable without loss of regularity and that the solution of the scattering problem is infinitely shape differentiable away from the boundary of the obstacle, whereas its derivatives lose regularity on the boundary. We also give a characterization of the first shape derivative as a solution of a new electromagnetic scattering problem.

**Conclusion.** This analysis has been used in the thesis [5] to develop a shape optimization algorithm of dielectric lenses in order to obtain a prescribed radiation pattern, with an implementation of the shape derivatives of the electromagnetic boundary integral operators in MATLAB. The shape derivative analysis of boundary integral operators between Sobolev spaces and their implementation are also new for the acoustic and the elastic scattering theory. This work offers many perspectives for the construction an the analysis of efficient method to solve shape optimization problems and inverse problems related to the scattering of time-harmonic waves.

**References**


