A Generalization of the Eckart and Young Theorem and the Proper Generalized Decomposition

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The Proper Generalized Decomposition (PGD) method has been recently proposed [1, 13, 16] for the a priori construction of separated representations of an element \( u \) in a tensor product space \( V = V_1 \otimes \ldots \otimes V_d \), which is the solution of a problem

\[
A(u) = l. \tag{1}
\]

A rank-\( n \) approximated separated representation \( u_n \) of \( u \) is defined by

\[
u_n = \sum_{i=1}^{n} v_i \otimes \ldots \otimes v_i \tag{2}\]

The a posteriori construction of such tensor decompositions, knowing the function \( u \), have been extensively studied over the past years in multilinear algebra community [5, 6, 11, 12, 4, 7] (essentially for finite dimensional vector spaces \( V_i \)). The question of finding an optimal decomposition of a given rank \( r \) is not trivial and has led to various definitions and associated algorithms for the separated representations.

In the context of problems of type (1), the solution is not known a priori, nor an approximation of it. An approximate solution is even unreachable with traditional numerical techniques when dealing with high dimensions \( d \). It is the so-called curse of dimensionality associated with the dramatic increase of the dimension of approximation spaces when increasing \( d \). The PGD method aims at constructing a decomposition of type (2) without knowing a priori the solution \( u \). The aim of the PGD is to construct a sequence \( u_n \) based on the knowledge of operator \( A \) and right-hand side \( l \). This can be achieved by introducing new definitions of optimal decompositions (2). The Proper Generalized Decomposition method have been first introduced under the name of “Radial-type approximation” for the solution of time dependent partial differential equations (PDE), by separating space and time variables, and used in the context of the LATIN method in computational solid mechanics [13, 9, 14, 20, 15]. It has been also introduced for the separation of coordinate in multidimensional PDEs [1, 2], with many applications in kinetic theory of complex fluids, financial mathematics, computational chemistry… It has also been introduced in the context of stochastic or parametrized PDEs by introducing a separation of physical variables (space, time…) and (random) parameters [16, 17, 18]. Still in the context of stochastic PDEs, a further separation of parameters have also been introduced, by exploiting the tensor product structure of stochastic function spaces [8, 19]. Several PGD definitions and associated algorithms have been proposed and have proved their efficiency in practical applications [17]. However, for most PGD definitions, their mathematical analysis remain open. In this paper, we investigate a particular case of PGD, which consists in defining the decomposition (2) progressively. This is a basic definition of the PGD which was proposed in [13, 16, 1]. A proof of convergence for this particular PGD has been introduced in [3] for the case of a second order elliptic symmetric partial differential equation defined in a 2-dimensional domain. Here, we consider the mathematical analysis of this PGD for a larger class of problems in an abstract setting. We introduce a generalization of Eckart and Young theorem [10] which allows to prove the convergence of progressive PGDs for a large class of linear problems defined in tensor product Hilbert spaces.
References


